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THE p-ADIC LOG GAMMA FUNCTION AND p-ADIC EULER CONSTANTS[1]

BY

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ABSTRACT. We define G_p, a p-adic analog of the classical log gamma function and show it satisfies relations similar to the standard formulas for log gamma. We also define p-adic Euler constants and use them to obtain results on G_p' and on the logarithmic derivative of Morita's Γ_p.

1. Introduction. Leopoldt and Kubota defined p-adic L-functions by summing a function of two variables with respect to one of the variables. We present a general theorem on this technique and then use it to define G_p, a p-adic analog of the classical log Γ function. We work with log Γ rather than Γ because the only continuous p-adic function defined on a subset of Ω_p and satisfying $f(x + 1) = xf(x)$ is the zero function. It is possible to construct an analog of Γ by modifying the functional equation (see Morita [7]), but then we do not have close analogs of the standard formulas for Γ or log Γ. For G_p, which is not the log of Morita's gamma function, we have the functional equation, an extension theorem, the Stirling series, the Gauss multiplication theorem, a power series, certain "Laurent" series and a formula due to Gauss which is valid for G_p' at rational points.

This last formula was discussed by Lehmer in [6], where he defined Euler constants for arithmetic progressions. We define p-adic Euler constants and present a proof of Gauss' theorem which is valid in both the p-adic and complex systems. We also apply the results on Euler constants to obtain a finite expression for the logarithmic derivative of Morita's p-adic gamma function at certain rational values in its domain.

2. Notation and definitions. We will use Q, Q_p, Z, Z_p, C and Ω_p for, respectively, the field of rational numbers, the p-adic completion of Q, the ring of rational integers, the p-adic completion of Z in Q_p, the field of complex numbers and the completion of the algebraic closure of Q_p. B_n will be the nth Bernoulli number defined by $te^t/(e^t - 1)$. ν will be the p-adic valuation on

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Ω_p with $\nu(p) = 1$ and $|\ |_p$ will be the absolute value on Ω_p with $|p|_p = p^{-1}$. We will use boldface letters to indicate r-tuples.

A polydisc about $\mathbf{c} \in \Omega_p^r$ is a set of the form

$$\{(x_1, \ldots, x_r): |x_i - c_i|_p \leq \rho_i, i = 1, 2, \ldots, r\}$$

where $\mathbf{c} = (c_1, \ldots, c_r)$ and all $\rho_i > 0$. $(\rho_1, \ldots, \rho_r)$ is called the radius of the polydisc. $\mathbf{a}$ and $\mathbf{M}$ will denote $(a_1, \ldots, a_r)$ and $(M_1, \ldots, M_r)$, respectively.

We call a function defined on a subset of Ω_p^r holomorphic if it can be represented by a single power series and locally holomorphic if at each point in the domain we can represent the function by a power series on some polydisc containing the point.

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3. **p-adic sums.** We begin by considering sums of the type used by Leopoldt and Kubota [5]. The first theorem is a generalization of a result in [2, p. 309].

THEOREM 1. *Suppose we have rational integers a_i, b_i, M_i with $a_i \geq 0$, $b_i \geq 1$, $M_i \geq 1$ for $i = 1, 2, \ldots, r$. Let R be an open set in Ω_p^r with $\mathbf{a} + \mathbf{M}Z_p \subset R$. B is a Banach space over Ω_p and $f: R \to B$ is locally holomorphic.*

We define

$$S(k_1, \ldots, k_r, b_1, \ldots, b_r) = \frac{1}{b_1 \ldots b_r p_1^{k_1} \ldots p_r^{k_r}} \sum_{i=1}^{r} \sum_{n_i = 0; n_i \equiv a_i \pmod{M_i}}^{M_i b_i p^{k_i} - 1} f(n_1, \ldots, n_r).$$

Then

(i) *$L = \lim_{\{k_i\} \to \infty} S(k_1, \ldots, b_r)$ exists;*

(ii) *L is independent of the $\{b_i\}$ used;*

(iii) *L may be calculated by iteration of the limit in any order.*

PROOF. We begin with $a_1, \ldots, a_r = 0$, $M_1, \ldots, M_r = 1$ and f holomorphic on R with $Z_p^r \subset R$. We can write

$$f(\mathbf{x}) = \sum_J a_J \mathbf{x}^J,$$

where the right side represents a power series in r variables with J running through the r-tuples of nonnegative integers.

After we substitute the series for f in the formula for S and use the fact (see [2]) that

$$\left| (-1)^j B_j - \frac{1}{bp^k} \sum_{n=0}^{bp^k - 1} n^j \right|_p < p^{2-k}$$

for $k = 0, 1, \ldots$ and $j = 0, 1, \ldots,$ it is easy to verify that

$$\lim_{k_r\to\infty} \cdots \lim_{k_1\to\infty} S(k_1, \ldots, b_r) = \sum_J a_J(-1)^J B_J,$$

and each limit is uniform with respect to the remaining variables. We can now conclude that

$$L = \lim_{\{k_i\}\to\infty} S(k_1, \ldots, b_r) \text{ exists.}$$

The next two parts of the theorem are obvious.

Now suppose f is locally holomorphic on R and $R \supset Z_p^r$. Then there is a finite covering of Z_p^r by polydiscs whereby f is holomorphic on each polydisc, each polydisc has the same radius and the radius has the form $(p^{-N}, \ldots, p^{-N})$. We let $A = \{0, \ldots, p^N - 1\}$ and for each $W \in A^r$ we define

$$f_W(\mathbf{x}) = p^{-Nr} f(W + p^N \mathbf{x}).$$

Each f_W is holomorphic on the disc with center $(0, \ldots, 0)$ and radius $(1, \ldots, 1)$.

It is convenient now to introduce an integral type notation.

We define

$$\int_{\mathbf{a},\mathbf{M}} f(\mathbf{x})\, d\mathbf{x} = L,$$

where L is defined in Theorem 1.

We have

$$\int_{\mathbf{0},\mathbf{1}} f(\mathbf{x})\, d\mathbf{x} = \sum_{W\in A^r} \int_{\mathbf{0},\mathbf{1}} f_W(\mathbf{x})\, d\mathbf{x},$$

and the conclusion of the theorem follows directly.

Finally, if $f\colon R \to B$ is locally holomorphic and $\mathbf{a} + \mathbf{M}Z_p \subset R$, we define $g(\mathbf{x}) = f(\mathbf{a}' + \mathbf{x}\mathbf{M})$ where $\mathbf{x}\mathbf{M} = (x_1M_1, \ldots, x_rM_r)$ and $\mathbf{a}'$ is the least nonnegative residue of $\mathbf{a}$ mod $\mathbf{M}$. Since g satisfies the conditions needed earlier in this proof, and $\int_{\mathbf{a},\mathbf{M}} f(\mathbf{x})d\mathbf{x} = \int_{\mathbf{0},\mathbf{1}} g(\mathbf{x})d\mathbf{x}$, we have established Theorem 1.

The next result is our basic device for constructing p-adic functions. We will use it to define a p-adic analog of $\log \Gamma$ and to define p-adic Euler constants. It can be used to show the existence and holomorphy of the p-adic L-functions and similarly constructed functions occurring in the works cited as references.

THEOREM 2. *Suppose a_i, M_i are rational integers with $a_i > 0$, $M_i > 1$ for $i = 1, 2, \ldots, r$, $\{C_1, \ldots, C_t\}$ is a set of polydiscs in Ω_p^r, $R = \bigcup_{i=1}^t C_i$ and $\mathbf{a} + \mathbf{M}Z_p \subset R$. Let D be a polydisc in Ω_p^s and suppose $f\colon R \times D \to \Omega_p$ is holomorphic on each $C_i \times D$, $i = 1, 2, \ldots, t$. Then*

$$F(\mathbf{x}) = \int_{\mathbf{a},\mathbf{M}} f(\mathbf{u}, \mathbf{x})\, d\mathbf{u}$$

exists and is holomorphic on D.

PROOF. We let $\Lambda(D)$ = Banach space of holomorphic functions from $D \to \Omega_p$. For $u \in R$ we define

$$\phi(\mathbf{u}) = \text{the mapping } \mathbf{x} \to f(\mathbf{u}, \mathbf{x}).$$

For a fixed $\mathbf{u}_i \in C_i$ we have

$$f(\mathbf{u}, \mathbf{x}) = \sum_J a_{i,J}(\mathbf{x})(\mathbf{u} - \mathbf{u}_i)^J$$

for all $\mathbf{u} \in C_i$ and $\mathbf{x} \in D$.

If $a_{i,J}$ denotes the map on D, $\mathbf{x} \to a_{i,J}(\mathbf{x})$, then

$$\phi(\mathbf{u}) = \sum_J a_{i,J}(\mathbf{u} - \mathbf{u}_i)^J$$

for $\mathbf{u} \in C_i$.

Each $a_{i,J} \in \Lambda(D)$, so ϕ: $R \to \Lambda(D)$ is locally holomorphic and we may apply Theorem 1. Since

$$\int_{\mathbf{a},\mathbf{M}} f(\mathbf{u}, \mathbf{x})\, d\mathbf{u} = \left(\int_{\mathbf{a},\mathbf{M}} \phi(\mathbf{u})\, d\mathbf{u}\right)(x),$$

we conclude that $F \in \Lambda(D)$.

The following corollary is a useful form of Theorem 2.

COROLLARY. *Suppose a, b, M are rational integers with $a > 0$, $b > 1$, $M > 1$. Let f be locally holomorphic on a set $A \subset \Omega_p$. Let $\mathbf{x} \in \Omega_p^s$ and $T(u, \mathbf{x})$ be locally holomorphic on some subset of Ω_p^{s+1}. Define $A^* = \{\mathbf{x} | T(a + MZ_p, \mathbf{x}) \subset A\}$.*

Then A^* is open, and if $A^* \neq \varnothing$,

$$F(\mathbf{x}) = \lim_{k\to\infty} \frac{1}{bp^k} \sum_{\substack{n=0 \\ n \equiv a \ (\mathrm{mod}\ M)}}^{Mbp^k - 1} f(T(n, \mathbf{x}))$$

is independent of b and locally holomorphic on A^*.

PROOF. Given $\mathbf{c} \in A^*$ and $u \in a + MZ_p$ there is a polydisc $D(u, \mathbf{c})$ containing $(u, \mathbf{c})$ on which $f \circ T$ is holomorphic. Holding $\mathbf{c}$ fixed, a finite number of $D(u_i, \mathbf{c})$ cover $(a + MZ_p, \mathbf{c})$. Each $D(u_i, \mathbf{c})$ has the form $C_i \times D_i$ where C_i is a disc in Ω_p containing u_i and D_i is a polydisc about $\mathbf{c}$. Let $D = \bigcap D_i$. We know the following:

(i) $T(C_i \times D) \subset A$ for each i, so $D \subset A^*$;

(ii) $\bigcup C_i$ covers $a + MZ_p$;

(iii) $f \circ T$ is holomorphic on each $C_i \times D$.

From Theorem 2 we see $F(\mathbf{x})$ is holomorphic on D and therefore locally holomorphic on A^*.

We will occasionally wish to differentiate $F(\mathbf{x})$. We have

THEOREM 3. *Using the definitions and conditions of Theorem* 2,

$$\frac{\partial F(\mathbf{x})}{\partial x_i} = \int_{\mathbf{a},\mathbf{M}} \frac{\partial f(\mathbf{u}, \mathbf{x})}{\partial x_i} \, d\mathbf{u}.$$

PROOF. We fix $\mathbf{x} \in D$ and for $t \in \Omega_p$ we let $t^* = (0, \ldots, t, \ldots, 0)$, t being in the ith position, $t^* \in \Omega_p^s$. We define

$$h(\mathbf{u}, t) = \frac{f(\mathbf{u}, \mathbf{x} + t^*) - f(u, x)}{t} \quad \text{for } t \neq 0$$

and $h(\mathbf{u}, 0) = \lim_{t \to 0} h(\mathbf{u}, t)$.

We observe that $h(\mathbf{u}, 0) = \partial f(\mathbf{u}, \mathbf{x})/\partial x_i$ and that there is a neighborhood D_0, of zero, so h is holomorphic on each $C_i \times D_0$.

From the definition of derivative we have

$$\frac{\partial F(\mathbf{x})}{\partial x_i} = \lim_{t \to 0} \int_{\mathbf{a},\mathbf{M}} h(\mathbf{u}, t) \, d\mathbf{u}.$$

Now, if we let $H(t) = \int_{\mathbf{a},\mathbf{M}} h(\mathbf{u}, t) \, d\mathbf{u}$ we can use Theorem 2 to see that H is holomorphic on D_0 and, in particular, continuous at 0.

Thus we have

$$\frac{\partial F(\mathbf{x})}{\partial x_i} = \lim_{t \to 0} H(t) = H(0) = \int_{\mathbf{a},\mathbf{M}} \frac{\partial f(\mathbf{u}, \mathbf{x})}{\partial x_i} \, d\mathbf{u}.$$

The next result shows how certain sums can be used to solve difference equations.

THEOREM 4. *If a, M are rational integers where $M > a \geq 0$, $f'(x + a)$ exists and $F(x) = \int_{a,M} f(x + u) \, du$, then $F(x + M)$ exists and $F(x + M) = F(x) + Mf'(x + a)$.*

PROOF. This follows directly from the definition of the right side.

4. The p-adic log Γ **function.** We now consider the problem of constructing a p-adic analog of $\log \Gamma(x)$.

In looking for a p-adic analog of $\log \Gamma(x)$ we want a function G_p which sends a subset of Ω_p into Ω_p and satisfies the functional equation $G_p(x + 1) = G_p(x) + \log(x)$. $\log(x)$ is defined by the usual power series when $|x - 1|_p < 1$, and by setting $\log(p) = 0$ and using the functional equations for $\log(x)$ when $|x - 1|_p \geq 1$ and $x \neq 0$. There is a complete discussion of this idea in [3]. Just as in the complex case, this functional equation forces G_p to be discontinuous on either the positive integers or the negative integers. This is somewhat unfortunate in the p-adic case because if we want a locally holomorphic function we must exclude Z_p from the domain of G_p. However, this is all we need exclude because on the domain $\Omega_p - Z_p$ we have a locally holomorphic function which satisfies $G_p(x+1) = G_p(x) + \log(x)$ and several other relations similar to those of the complex $\log \Gamma(x)$. We will use the construction given in §3 to define $G_p(x)$ and demonstrate its properties.

An alternative approach is to slightly modify the functional equation to obtain a functional locally holomorphic on all of Ω_p. After considering $G_p(x)$ we will exhibit a sequence of such functions. We have the relation that the sequence of functions locally holomorphic on Ω_p converges pointwise to G_p.

The technique of changing the functional equation has been used by Morita [7] to define Γ_p, a function on Z_p, which is an analog of Γ. Our G_p is clearly not $\log \Gamma_p$.

DEFINITION OF G_p. We use the corollary of Theorem 2 with $T(u, x) = u + x$ and $f(x) = x\log(x) - x$. f is locally holomorphic on $\Omega_p - \{0\}$. We then have

$$G_p(x) = \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} (x+n)\log(x+n) - (x+n).$$

G_p is locally holomorphic on $\Omega_p - Z_p$, and at each $c \in \Omega_p - Z_p$ the disc of holomorphy is the largest (open) disc $D(c)$ such that $D(c) \cap Z_p = \varnothing$.

An immediate consequence of Theorem 4 is the functional equation:

THEOREM 5. $G_p(x+1) = G_p(x) + \log x$.

Stirling's Theorem, which is an asymptotic formula for $\log \Gamma(x)$, is simpler in Ω_p. We have

THEOREM 6. *When* $|x|_p > 1$,

$$G_p(x) = \left(x - \tfrac{1}{2}\right) \log(x) - x + \sum_{r=1}^{\infty} \frac{B_{r+1}}{r(r+1)x^r}.$$

PROOF.

$$G_p(x) = \frac{1}{2} - x + \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} (x+n)(\log(x) + \log(1+n/x)).$$

Using the power series for $\log(1+n/x)$ will lead to the result.

If we match Theorem 6 and the next result with the corresponding classical formulas we see that it is more accurate to speak of $G_p(x)$ as the analog of $-\frac{1}{2}\log(2\pi) + \log \Gamma(x)$. However, for simplicity we will continue to refer to G_p as the analog of $\log \Gamma$.

The following relation is the p-adic version of Gauss' Multiplication Theorem.

THEOREM 7. *Given any* $m \in Z^+$ *we have*

$$G_p(x) = \left(x - \tfrac{1}{2}\right)\log(m) + \sum_{a=0}^{m-1} G_p\left(\frac{x+a}{m}\right)$$

provided the right side is defined.

PROOF. We can write

$$G_p(x) = \lim_{k\to\infty} \frac{1}{mp^k} \sum_{n=0}^{mp^k-1} (x+n)\log(x+n) - (x+n)$$

$$= \lim_{k\to\infty} \frac{1}{mp^k} \sum_{n=0}^{p^k-1} \sum_{a=1}^{m-1} (x+a+mn)\log(x+a+mn) - (x+a+mn).$$

With a little rearranging, Theorem 7 is easily obtained.

COROLLARY.

$$G_p(x) = \sum_{a=0}^{p^r-1} G_p\left(\frac{x+a}{p^r}\right) \quad \textit{for } r = 0, 1, 2, \ldots.$$

This last corollary provides us with a means for transferring results about $G_p(x)$ when $|x|_p > 1$ to $G_p(x)$ with $|x|_p \leqslant 1$.

For the extension theorem we have

THEOREM 8. $G_p(x) + G_p(1-x) = 0$.

PROOF. We can see immediately from Theorem 6 that $G_p(x) + G_p(-x) = -\log(x)$ when $|x|_p > 1$. Combining this with $G_p(x+1) = G_p(x) + \log(x)$ we have $G_p(x) + G_p(1-x) = 0$ when $|x|_p > 1$.

Given any $x \in \Omega_p - Z_p$ with $|x|_p \leqslant 1$ we can choose an $r \in Z^+$ so $|(x+a)/p^r|_p > 1$ for all $a \in Z$. Then

$$G_p(x) + G_p(1-x) = \sum_{a=0}^{p^r-1} G_p\left(\frac{x+a}{p^r}\right) + G_p\left(\frac{1-x+a}{p^r}\right)$$

$$= \sum_{a=0}^{p^r-1} G_p\left(\frac{x+a}{p^r}\right) - G_p\left(1 - \frac{1-x+a}{p^r}\right)$$

$$= \sum_{a=0}^{p^r-1} G_p\left(\frac{x+a}{p^r}\right) - G_p\left(\frac{x+p^r-a-1}{p^r}\right),$$

and, since as a goes from 0 to $p^r - 1$, $p^r - a - 1$ goes from $p^r - 1$ to 0, Theorem 8 is proven.

The complex $\log \Gamma(x)$ has a simple power series about 1, with values of the Riemann ζ-function appearing in the coefficients. We will now find the power series for $G_p(x)$ about $1/p$.

We use Theorem 4 to obtain

$$D^{(1)}G_p(x) = \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \log(x+n)$$

and, in particular,

$$D^{(1)}G_p(1/p) = p \lim_{k\to\infty} \frac{1}{p^k} \sum_{\substack{m=0 \\ m\equiv 1 \pmod p}}^{p^k-1} \log(m).$$

We write this as

$$D^{(1)}G_p(1/p) = -p\gamma_p(1,p),$$

$$D^{(r)}G_p(x) = (-1)^r(r-2)! \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \frac{1}{(x+n)^{r-1}} \quad \text{for } r \geq 2,$$

$$\frac{D^{(r)}G_p(1/p)}{r!} = \frac{(-1)^r}{r(r-1)} \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \frac{1}{(n+1/p)^{r-1}}$$

$$= \frac{(-1)^r p^r}{r} \cdot \frac{1}{r-1} \lim_{k\to\infty} \frac{1}{p^k} \sum_{\substack{m=0 \\ m\equiv 1 \pmod p}}^{p^k-1} \frac{1}{m^{r-1}}.$$

We will write this last expression as $(-1)^r p^r \zeta_p(r)/r$. Using the notation introduced above we have

THEOREM 9.

$$G_p(x) = G_p\left(\frac{1}{p}\right) - p\gamma_p(1,p)\left(x - \frac{1}{p}\right) + \sum_{r=2}^{\infty} \frac{(-1)^r \zeta_p(r) p^r}{r} \left(x - \frac{1}{p}\right)^r.$$

This series converges for $|x - 1/p|_p < p$.

It is interesting to compare this with the classical formula for $\log \Gamma(x)$:

$$\log \Gamma(x) = -\gamma(x-1) + \sum_{r=2}^{\infty} \frac{(-1)^r \zeta(r)}{r}(x-1)^r \quad \text{for } |x-1| < 1.$$

The idea of having $p = 1$ give us classical results from a p-adic formula, while only formal here, is valid in certain formulas for $\zeta(n)$ and $\zeta_p(n)$ discussed in [1].

Our next result for $G_p(x)$ is a set of formulas for $G_p(x)$ valid on the annular regions $A_n = \{x: n-1 < \nu(x) < n\}$ for $n = 1, 2, 3, \ldots$. Since these regions have no points of Z_p we are able to find series which are almost Laurent series. To simplify the discussion we introduce a function G^* defined by

$$G^*(x) = \lim_{k\to\infty} \frac{1}{p^k} \sum_{\substack{n=0 \\ p\nmid n}}^{p^k-1} f(x+n)$$

where $f(x) = x\log(x) - x$.

We can write

$$G^*(x) = \sum_{a=1}^{p-1} \lim_{k\to\infty} \frac{1}{p^k} \sum_{\substack{n=0 \\ n\equiv a \pmod p}}^{p^k-1} f(x+n).$$

For each value of a the inner lim is locally analytic for x with $x + a \notin pZ_p$. Therefore G^* is locally analytic for $x \in \Omega_p - V_p$, where V_p is the set of units in Z_p.

For $|x|_p < 1$, G^* coincides with a function defined by Morita [7] in the study of the function he calls $\Gamma_p(x)$.

To obtain our formulas for G_p we need the power series for $G^*(x)$ at $x = 0$.

THEOREM 10. *If* $|x|_p < 1$, *then*

$$G^*(x) = M_{\varepsilon_p}(\log)(x) + \sum_{r=3}^{\infty} \frac{L_p(r, \bar{\omega}^{r-1})(-1)^r}{r} x^r.$$

$L_p(r,\chi)$ *is the Leopoldt L-function for the character* χ, ε_p *is the principal character* mod p, $M_\chi(f)$ *is the Leopoldt* χ*-mean* [5], *and* $\bar{\omega}$ *is the character* mod p *defined by*

$$\bar{\omega}(n) = \begin{cases} \lim\limits_{k\to\infty} n^{-p^k} & \text{for } (n,p) = 1, \text{if } p > 2, \\ 1 & \text{if } n \equiv 1 \bmod 4, \\ -1 & \text{if } n \equiv 3 \bmod 4 \text{ for } p = 2. \end{cases}$$

PROOF. Apply Theorem 3.

This result has also been found by Morita [7].

We are now prepared to find series for $G_p(x)$ on the annular domains $A_n = \{x: n - 1 < \nu(x) < n\}, n \in Z^+$.

With $x \in A_n$ we write the equations

$$G_p(x/p^i) - G_p(x/p^{i+1}) = G^*(x/p^i) \quad \text{for } i = 0, 1, \dots, n-1;$$

adding these equations we obtain

$$G_p(x) = G_p\left(\frac{x}{p^n}\right) + \sum_{i=0}^{n-1} G^*\left(\frac{x}{p^i}\right).$$

We now use Theorems 6 and 10 to obtain

THEOREM 11. *On the annulus* A_n *we have the formula*

$$G_p(x) = \left(\frac{x}{p^n} - \frac{1}{2}\right)\log(x) - \frac{x}{p^n} + M_{\varepsilon_p}(\log)\left(\frac{p^n - 1}{p^n - p^{n-1}}\right)x$$
$$+ \sum_{r=3}^{\infty} \frac{L_p(r, \bar{\omega}^{r-1})(-1)^r}{r}\left(\frac{p^{rn} - 1}{p^{rn} - p^{r(n-1)}}\right)x^r + \sum_{r=1}^{\infty} \frac{B_{r+1}p^{nr}x^{-r}}{r(r+1)}.$$

If we define A_0 *as* $\{x: |x|_p > 1\}$, *then the above formula is valid for* $n = 0, 1, 2, \ldots$.

5. **Analyticity.** The function G_p is not an analytic functionin the sense of Krasner [4], but its second derivative G_p'' is an analytic function on $\Omega_p - Z_p$.

THEOREM 12. *G_p'' is an analytic function on* $\Omega_p - Z_p$.

PROOF. First for $a, m \in Z$ we define $D(a, m) = \{x: x \in \Omega_p, \nu(x - a) \geq m\}$. For each $m \in Z^+$ the set $A_m = \Omega_p - \bigcup_{a=0}^{p^n-1} D(a, m)$ is a quasi-connected set. $\{A_m: m \in Z^+\}$ is nested and $\bigcup_{m=1}^{\infty} A_m = \Omega_p - Z_p$. Therefore if we can prove G_p'' is an analytic element on each A_m, i.e. the uniform limit of a sequence of rational functions having no poles in A_m, then we will know G_p'' is an analytic function on $\Omega_p - Z_p$.

If we apply Theorem 7 we can write

$$G_p''(x) = \frac{1}{p^{2m+2}} \sum_{a=0}^{p^{m+1}-1} G_p''\left(\frac{x+a}{p^{m+1}}\right)$$

for each $m \in Z^+$ and $x \in \Omega_p - Z_p$.

If we consider just $x \in A_m$, then $|(x + a)/p^{m+1}|_p > p > 1$ for all $a \in Z$. Therefore we may use Theorem 6 and obtain

$$G_p''\left(\frac{x+a}{p^{m+1}}\right) = \sum_{r=0}^{\infty} \frac{B_r}{\left[(x+a)/p^{m+1}\right]^{r+1}}.$$

Since $|(x + a)/p^{m+1}|_p > p$ for all $x \in A_m$, this last series converges uniformly on A_m.

Thus G_p'' is an analytic element on A_m and G_p'' is an analytic function on $\Omega_p - Z_p$.

6. **An alternative approach.** Earlier we mentioned another approach to the idea of p-adic $\log \Gamma(x)$: to change the functional equation. Of course, it must only be a slight change so we can associate it with $\log \Gamma(x)$. We will construct a sequence of such functions, which will be locally holomorphic on Ω_p and have $G_p(x)$ as their pointwise limit.

DEFINITION. Let

$$H_N(x) = \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} f_N(x+n) \quad \text{for } N = 1, 2, \ldots,$$

where

$$f_N(x) = \begin{cases} x\log(x) - x & \text{if } \nu(x) < N, \\ 0 & \text{if } \nu(x) \geq N. \end{cases}$$

Each f_N is locally analytic on Ω_p, so each H_N is also locally analytic on Ω_p.

We have the following equation

$$H_N(x+1) = \begin{cases} H_N(x) + \log(x) & \text{if } \nu(x) < N, \\ H_N(x) & \text{if } \nu(x) \geqslant N. \end{cases}$$

It can be shown that $H_N(0) = 0$ for each N, so we have for $n \in Z^+$,

$$H_N(n+1) = \log \prod_{\substack{t=1 \\ p^N \nmid t}}^{n} t,$$

in particular, $H_N(n+1) = \log(n!)$ if $n < p^N$. The following theorem shows the relation between $G_p(x)$ and $H_N(x)$.

THEOREM 13. *If x is such that $|x - a|_p > p^{-N}$ for all $a \in Z_p$ then $H_N(x) = G_p(x)$.*

PROOF. Inspection of the definitions of $G_p(x)$ and $H_N(x)$.

Theorem 13 shows us that the sequence $H_N(x)$, $N = 1, 2, \ldots,$ and x fixed with $x \notin Z_p$, eventually becomes constant with the value $G_p(x)$.

For x with $\nu(x) \geqslant 1$ the functions H_1 and G^* coincide. However, for other x they are not the same and it is H_1 which is the log of the function on Z_p which Morita has called $\Gamma_p(x)$ [7].

7. ***p*-adic Euler constants.** In a recent paper [6], D. H. Lehmer proves a theorem of Gauss by defining a generalization of Euler's constant. Gauss' theorem is a formula for the logarithmic derivative of the Gamma function at rational points r/k with $0 < r < k$. The formula is notable because it is a constant plus a linear combination of logarithms of integers in $Q(\sqrt[k]{1})$.

We shall define p-adic Euler constants, give the basic results for them and then prove the p-adic version of Gauss' theorem: a formula for $G_p'(r/f)$ with $0 < r < f$ and $\nu(r/f) < 0$.

We show how Gauss' theorem follows from the classical formula for $L(1, \chi)$, and since we have the same expression for $L_p(1, \chi)$ [3], we have a proof valid in both C and Ω_p.

Lehmer defines the (generalized) Euler constants by

$$\gamma(r, k) = \lim_{x\to\infty} \left[\sum_{\substack{0<n\leqslant x \\ n\equiv r \ (\mathrm{mod}\ k)}} \frac{1}{n} - \frac{1}{k} \log x \right]$$

and then, using the relation

$$\psi(r/k) = D^{(1)} \log \Gamma(r/k) = \log k - k\gamma(r, k) \quad \text{for } r, k \in Z^+, r \leqslant k,$$

he proves: If $r, k \in Z^+$, $r \leqslant k$, then

$$\psi\left(\frac{r}{k}\right) = -\gamma - \log\left(\frac{k}{2}\right) - \frac{\pi}{2}\cot\left(\frac{\pi r}{k}\right)$$
$$+ 2 \sum_{0<j<k/2} \cos\left(\frac{2\pi rj}{k}\right)\log\sin\left(\frac{\pi j}{k}\right).$$

Gauss' theorem, combined with the functional equation, enables us to calculate $\psi(x)$ in closed form at every rational value of x for which the function is defined.

Working in Ω_p we can define $\gamma_p(r,f)$ when $r, f \in Z, f > 1$, and find a similar formula for $G_p'(r/f)$.

When $\nu(r/f) < 0$ we define

$$\gamma_p(r,f) = -\lim_{k\to\infty} \frac{1}{fp^k} \sum_{\substack{m=0 \\ m\equiv r \,(\mathrm{mod}\, f)}}^{fp^k-1} \log(m).$$

When $\nu(r/f) \geq 0$ we write $f = p^k f^*$ with $(p, f^*) = 1$ and let $\phi = \phi(f^*)$ (the Euler function). We then define

$$\gamma_p(r,f) = \frac{p^\phi}{p^\phi - 1} \sum_{n\in N(r,f)} \gamma_p(r + nf, p^\phi f)$$

where

$$N(r,f) = \left\{n{:}0 \leq n < p^\phi, nf + r \not\equiv 0 \,(\mathrm{mod}\, p^{\phi+k})\right\}.$$

Theorem 1 applies to show $\gamma_p(r,f)$ exists.

To obtain Gauss' theorem in Ω_p we need several results which are mostly the same as Lehmer has given for C. The proofs follow from the definition of $\gamma_p(r,f)$, previous results for $G_p'(x)$ and Theorem 18. We will write $\psi_p(x) = G_p'(x)$.

THEOREM 14. (i) *If* $d|(r,f)$, *then* $f\gamma_p(r,f) = (f/d)\gamma_p(r/d, f/d) - \log d$.
(ii) *If* $\nu(r/f) < 0$ *and* $0 < r < f$, *then* $\psi_p(r/f) = -\log f - f\gamma_p(r,f)$.
(iii) $\gamma_p(r,f) = \gamma_p(f - r, f)$.
(iv) *If* $b \in Z^+$, *then*

$$\gamma_p(r,f) = \sum_{n=0}^{b-1} \gamma_p(r + nf, bf).$$

(v) *If* $p^\mu \equiv 1 \,(\mathrm{mod}\, f^*)$ *and* $\nu(r/f) \geq 0$, *then*

$$\gamma_p(r,f) = \frac{p^\mu}{p^\mu - 1} \sum_{\substack{n=0 \\ nf+r\not\equiv 0 \,(\mathrm{mod}\, p^{\mu+k})}}^{p^\mu-1} \gamma_p(r + nf, p^\mu f).$$

We are going to need a p-adic analog of Euler's constant. The value

$$\gamma_p = \gamma_p(0, 1) = -\frac{p}{p-1} \lim_{k\to\infty} \frac{1}{p^k} \sum_{\substack{m=1\\(m,p)=1}}^{p^k-1} \log(m)$$

fits our formulas precisely as we need. Morita has also realized the connection between Euler's constant and $M_{\mathfrak{p}}(\log)$ and he gives a slightly different value [7].

Lehmer has defined the formula $\Phi(f)$ by

$$\Phi(f) = \sum_{\substack{r=1\\(r,f)=1}}^{f} \gamma(r, f).$$

He then proves

$$\Phi(f) = \frac{\phi(f)}{f}\gamma + \frac{\phi(f)}{f}\sum_{q|f}\frac{\log q}{q-1}.$$

In this formula, q is prime, ϕ is the Euler ϕ-function and γ is Euler's constant.

We define

$$\Phi_p(f) = \sum_{\substack{r=1\\(r,f)=1}}^{f} \gamma_p(r, f) \quad \text{when } \nu(f) > 0.$$

We then have

THEOREM 15.

$$\Phi_p(f) = \frac{\phi(f)}{f}\gamma_p + \frac{\phi(f)}{f}\sum_{q|f}\frac{\log q}{q-1}.$$

PROOF. We can use Theorem 14(iv) with $b = f_1/f$ to show that if f has the same distinct prime factors as f_1 and $f|f_1$ then $\Phi_p(f) = \Phi_p(f_1)$.

It is then sufficient to consider square free f. This is accomplished by induction on the number of prime factors of f.

We will need the following algebraic identity.

THEOREM 16. *If ζ is a primitive fth root of unity, $f > 1$, ε_f the principal character* mod f *and $\tau_a(\varepsilon_f)$ the Gauss sum,*

$$\tau_a(\varepsilon_f) = \sum_{n=1}^{f} \varepsilon_f(n)\zeta^{an},$$

then

$$\prod_{a=1}^{f-1} (1 - \zeta^{-a})^{\tau_a(\varepsilon_f)} = \prod_{q|f} q^{-\phi(f)/(q-1)}.$$

The product on the right side is over the distinct prime divisors of f.

PROOF. Let ω_r be a primitive rth root of unity and $Q_r = Q(\omega_r)$ for $r = 2, 3, \ldots$.

We observe that

$$\tau_a(\varepsilon_f) = \sum_{\substack{n=1 \\ (n,f)=1}}^{f-1} \zeta^{an} = \mathrm{tr}_{Q_f/Q}(\zeta^a).$$

When we group together conjugate elements we have

$$\prod_{a=1}^{f-1} (1 - \zeta^{-a})^{\tau_a(\varepsilon_f)} = \prod_{r|f} \left(N_{Q_r/Q}(1 - \omega_r)\right)^{\mathrm{tr}_{Q_f/Q}(\omega_r)}. \tag{1}$$

Examination of the minimal polynomial of ω_r shows that:

(i) if r is not square free, then $\mathrm{tr}_{Q_f/Q}(\omega_r) = 0$;

(ii) if r is square free, but not prime, then $N_{Q_r/Q}(1 - \omega_r) = 1$;

(iii) if r is prime, $N_{Q_r/Q}(1 - \omega_r) = r$;

(iv) if r is prime

$$\mathrm{tr}_{Q_f/Q}(\omega_r) = \frac{\phi(f)}{r-1}\mathrm{tr}_{Q_r/Q}(\omega_r) = \frac{-\phi(f)}{r-1}.$$

Placing these four values into (1) establishes the theorem.

Now we state Gauss' theorem in Ω_p.

THEOREM 17. *If* $r, f \in Z^+$, $r < f$ *and* $\nu(r/f) < 0$, *then*

$$\psi_p(r/f) = -\log f - \gamma_p + \sum_{a=1}^{f-1} \zeta^{-ar}\log(1 - \zeta^a).$$

If ψ_p is replaced by ψ and γ_p by γ we have Gauss' theorem in C. Of course log is either p-adic or complex as required.

PROOF. Since we have shown (Theorem 14(ii)) that $\psi_p(r/f) = -\log f - f\gamma_p(r, f)$, it will be sufficient to prove

THEOREM 18. *If* $f > 1$ *and* ζ *is a primitive* f*th root of unity, then*

$$f\gamma_p(r, f) = \gamma_p - \sum_{a=1}^{f-1} \zeta^{-ar}\log(1 - \zeta^a).$$

Notice that we do not need the restriction $\nu(r/f) < 0$ *for this result.*

PROOF. We begin by assuming $(r, f) = 1$ and $\nu(r/f) < 0$ and proceed to evaluate $\sum_{\chi \neq \varepsilon_f} \bar{\chi}(r) L_p(1, \chi)$ in two different ways. The sum is over all non-principal characters mod f.

For χ not principal and if $p|f$, we have the forumla [3]

$$L_p(1, \chi) = -\frac{1}{f}\sum_{a=1}^{f-1} \tau_a(\chi)\log(1 - \zeta^{-a}).$$

(NOTE: Iwasawa gives this formula in a form valid only for primitive characters, but if $\bar{\chi}(a)\tau(\chi)$ is replaced by $\tau_a(\chi)$, then his proof can be modified to be valid for all nonprincipal characters.)

This is the same as the formula for $L(1, \chi)$ in C. Using this result and Theorems 15 and 16 we have

$$(*)\quad \sum_{\chi \neq \varepsilon_f} \bar{\chi}(r) L_p(1, \chi) = \frac{\phi(f)\gamma_p}{f} - \Phi_p(f) - \frac{\phi(f)}{f} \sum_{a=1}^{f-1} \zeta^{ar} \log(1 - \zeta^{-a}).$$

This result is also valid, correctly interpreted, in C.

If we use the expression [5] that

$$L_p(1, \chi) = -\lim_{k \to \infty} \frac{1}{fp^k} \sum_{n=0}^{fp^k - 1} \chi(n) \log n \quad \text{where } (n, p) = 1,$$

we obtain

$$(**)\qquad \sum_{\chi \neq \varepsilon_f} \bar{\chi}(r) L_p(1, \chi) = \phi(f)\gamma_p(r, f) - \Phi_p(f).$$

(**) is obtained in C by using

$$L(1, \chi) = \lim_{x \to \infty} \sum_{0 < n < x} \frac{\chi(n)}{n}.$$

Equating (*) and (**) yields Theorem 18 in the case where $(r, f) = 1$ and $\nu(r/f) < 0$.

Now suppose $(r, f) = d > 1$ and $\nu(r/f) < 0$. Then we can use 14(i) to obtain

$$f\gamma_p(r, f) = -\log d + \gamma_p - \sum_{a=1}^{f/d-1} \zeta^{-ar} \log(1 - \zeta^{ad}).$$

We can factor $1 - \zeta^{ad}$ and obtain

$$f\gamma_p(r, f) = -\log d + \gamma_p - \sum_{a=1}^{f/d-1} \sum_{b=0}^{d-1} (\zeta^a \lambda^b)^{-r} \log(1 - \zeta^a \lambda^b)$$

where λ is a primitive dth root of unity.

Since

$$\{\zeta^a \lambda^b : 0 \leqslant a < f/d, 0 \leqslant b \leqslant d-1\} = \{\zeta^a : 0 \leqslant a < f\},$$

$$f\gamma_p(r, f) = -\log d + \gamma_p - \sum_{a=1}^{f-1} \zeta^{-ar} \log(1 - \zeta^a) + \sum_{b=1}^{d-1} \log(1 - \lambda^b)$$

$$= \gamma_p - \sum_{a=1}^{f-1} \zeta^{-ar} \log(1 - \zeta^a).$$

This completes the proof of Gauss' theorem, but we have not yet completed the proof of Theorem 18.

If $\nu(r,f) > 0$ we can use the definition of $\gamma_p(r,f)$ and the case of Theorem 18 already proven to show

$$f\gamma_p(r,f) = \gamma_p - \frac{1}{p^\phi - 1} \sum_{a=1}^{p^\phi f-1} \eta^{-ar}\log(1-\eta^a) \sum_{n \in N(r,f)} \eta^{-anf},$$

where η is a primitive $p^\phi f$ root of unity.

The last sum on the right is $p^\phi - 1$ if $p^\phi | a$ and $-\eta^{ar-arp^\phi}$ if $p^\phi \nmid a$. Thus

$$f\gamma_p(r,f) = \gamma_p - \sum_{a=1}^{f-1} \zeta^{-ar}\log(1-\zeta^a) + \frac{1}{p^\phi - 1} \sum_{\substack{a=1 \\ p^\phi \nmid a}}^{p^\phi f-1} \zeta^{-ar}\log(1-\eta^a).$$

The last sum on the right is 0. □

We have seen that ψ_p is locally holomorphic on $\Omega_p - Z_p$ and ψ_p' is Krasner-analytic on this domain. We have also shown that the formula

$$(*) \qquad -\log f - f\gamma_p(r,f)$$

depends only on the ratio r/f and that for $\nu(r/f) < 0$, and $0 < r < f$,

$$\psi_p(r/f) = (*) = -\log f - \gamma_p + \sum_{a=1}^{f-1} \zeta^{-ar}\log(1-\zeta^a).$$

Since $(*)$ is defined for r, f with $\nu(r/f) > 0$, it is tempting to use $(*)$ to extend the definition of ψ_p onto the rational numbers in Z_p. However, this "continuation" would not retain the other properties of ψ_p. The values of $(*)$, though, are related to functions similar to ψ_p when $\nu(r,f) > 0$ and we have

THEOREM 19. *Given $\nu(r/f) > 0$, then for any μ such that $p^\mu \equiv 1 \pmod{f^*}$ we have*

$$(*) = \frac{p^\mu}{p^\mu - 1} H_\mu'\left(\frac{r}{f}\right) = -\log f - \gamma_p + \sum_{a=1}^{f-1} \zeta^{-ar}\log(1-\zeta^a).$$

H_N is discussed at the end of §4.

PROOF. This follows directly from previous results.

Since H_1 (on Z_p) is the logarithm of Morita's Γ_p [7], we have a

COROLLARY. *If $0 < r < f$, $\nu(r/f) > 0$ and $f^* | (p-1)$, then*

$$\frac{\Gamma_p'}{\Gamma_p}\left(\frac{r}{f}\right) = (1 - 1/p)\left(-\log f - \gamma_p + \sum_{a=1}^{f-1} \zeta^{-ar}\log(1-\zeta^a)\right).$$

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